An Introduction to Character Theory

J.R. McHugh

March 21, 2018

References

- *Abstract Algebra*, Third Edition. David S. Dummit and Richard M. Foote. 2004 John Wiley and Sons, Inc. (See chapters 18 and 19.)
- *Character Theory of Finite Groups*. I. Martin Isaacs. 1976 Academic Press.
- *A Course in Finite Group Representation Theory*. Peter Webb. 2016 Cambridge University Press. (A pre-publication version may be found at Peter Webb's site `http://www-users.math.umn.edu/~webb/`)

Motivation

Consider the set D_8 whose elements are the symmetries of a square.

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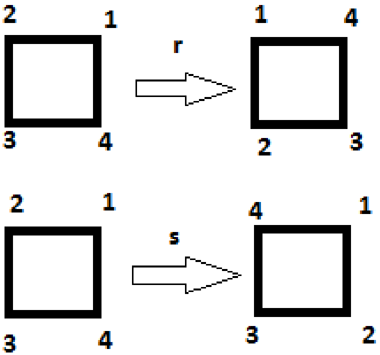
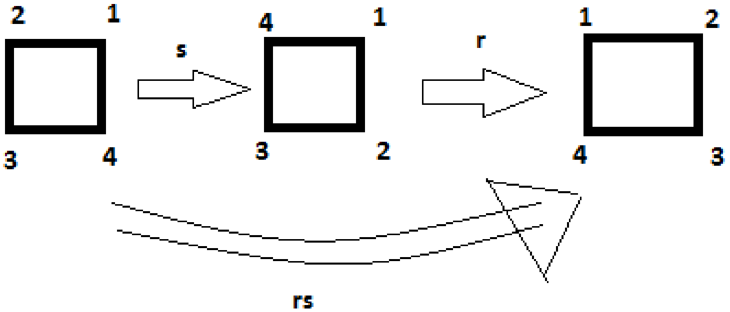
Two specific symmetries r and s in D_8 are described by the diagrams below:



We define a multiplication in the set D_8 by the rule: if x and y are symmetries then xy is the symmetry obtained by first performing y and then performing x.

For example, the symmetry rs is described by:



The multiplication in D_8 we have defined is associative and D_8 has an identity element 1 and inverses. In other words, the set D_8 together with our multiplication is a *group*.

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It is not difficult to see that every symmetry in D_8 is equal to a product of the form $r^i s^j$. We say that the group D_8 is *generated* by r and s.

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The symmetries r and s satisfy $r^4 = 1$, $s^2 = 1$, and $srs = r^{-1}$.

It turns out that these equalities (called *relations*) completely determine how symmetries multiply together in D_8. We thus obtain a *presentation* for the group D_8:

$$D_8 = \langle r, s | r^4 = s^2 = 1, srs = r^{-1} \rangle.$$

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If we begin with the abstract definition of D_8 as a group, how can we recover the square-symmetry interpretation for its elements?

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If we begin with the abstract definition of D_8 as a group, how can we recover the square-symmetry interpretation for its elements?

Here's one way. Begin by noting that a symmetry of the square permutes its vertices and is uniquely determined by how it permutes the vertices. We then have a one-to-one map

$$X : D_8 \to S_4.$$

(e.g., $X(r) = (1\,2\,3\,4)$ and $X(s) = (2\,4)$.) The map X is known as a *permutation representation* and is in fact a group homomorphism.

Another way is to use a *group action*.

Definition

Let G be a group and Ω a set. A *group action* of G on Ω is a map from $G \times \Omega$ to Ω written $g \cdot \omega$ for $g \in G$, $\omega \in \Omega$ that satisfies both $g \cdot (h \cdot \omega) = gh \cdot \omega$ and $1 \cdot \omega = \omega$ for all $g, h \in G$ and $\omega \in \Omega$.

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When a group action of G on Ω exists, we say G *acts on* Ω. For example, if V is a vector space over the field F then the multiplicative group $F^\times = F \setminus \{0\}$ acts on V since $c \cdot (d \cdot v) = cd \cdot v$ and $1 \cdot v = v$ for all $c, d \in F$ and $v \in V$ (these are axioms built into the definition of a vector space).

Another example: any group G acts on itself by left multiplication.

Let $X : D_8 \to S_4$ be the permutation representation of D_8 from before. We can define a group action of D_8 on $\Omega = \{1, 2, 3, 4\}$ by $g \cdot \omega = X(g)(\omega)$ for $g \in D_8$ and $\omega \in \Omega$. For example, $r \cdot 1 = 2$ and $s \cdot 2 = 4$.

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The fact that this defines an actual group action hinges on the fact that X is a homomorphism:

$$g \cdot (h \cdot \omega) = X(g)(X(h)(\omega)) = (X(g) \circ X(h))(\omega) = X(gh)(\omega) = gh \cdot \omega$$

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We can interpret each element of D_8 as a symmetry of the square via this group action.

We can recover the permutation representation X from the action of D_8 on $\Omega = \{1, 2, 3, 4\}$.

The map from Ω to itself induced by left multiplication by an element $g \in D_8$ has a 2-sided inverse (can you guess what the inverse is?). In other words this map is a permutation, so we can treat it as an element of S_4.

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The map $X : D_8 \to S_4$ defined by $X(g) = (\omega \mapsto g \cdot \omega)$ is a homomorphism because

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The group D_8 is not special: in general, a group action determines a permutation representation and vice versa.

To recap what we have done: we began with a "concrete" group (D_8) with an action "built into" the group itself (the elements of D_8 were symmetries of a square). We then split the concrete group up into two components: an "abstract" group (D_8 defined by generators and relations, i.e., a set of symbols with multiplication) and a group action/permutation representation. What is the use of this latter perspective?

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Both of these theorems are proved using the abstract theory of group actions.

Recall that the *general linear group of degree n over the field F* is the group $GL_n(F)$ of invertible $n \times n$ matrices with entries in the field F. If V is a vector space over F, the group $GL(V)$ is the group of nonsingular linear transformations of V.

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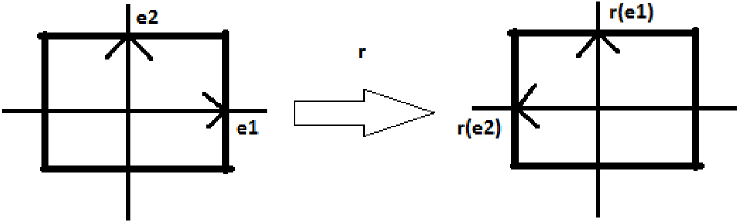
If we embed the square at the center of a Cartesian coordinate system we can reinterpret the action of D_8 on the square via linear transformations:



In other words, the action of D_8 on the square can be thought of as a homomorphism $X : D_8 \to GL_2(\mathbb{R})$ with

$$X(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } X(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Representations

Definition

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In case $\dim(V) = n < \infty$ we obtain an isomorphism $GL(V) \cong GL_n(F)$ after choosing a basis for V. A homomorphism $X : G \to GL_n(F)$ is sometimes called a *matrix representation*. The *degree* of the representation is n.

Just as for permutation representations, we may view a representation $X : G \to GL(V)$ as a description of an action of G on the vectors in V — simply define $g \cdot v = X(g)(v)$. The action of G on V satisfies, for all $g, h \in G$, $v, w \in V$, and $\lambda \in F$,

(1) $g \cdot (h \cdot v) = gh \cdot v$

(2) $1 \cdot v = v$

(3) $g \cdot (v + \lambda w) = g \cdot v + \lambda(g \cdot w)$.

An action of G on an F-vector space which satisfies all of the above is called a *linear action*.

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An action of G on an F-vector space which satisfies all of the above is called a *linear action*.

Given a linear action of G on V we may define a representation $X : G \to GL(V)$ by letting $X(g)$ denote the linear operator $v \mapsto g \cdot v$.

Some examples:

- We can create a homomorphism from any group G to $GL(V)$ by defining $X(g) = 1$ for all $g \in G$. This representation is called the *trivial representation*.

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- Let $e_1, e_2, \ldots, e_n$ be a basis for the vector space V over F. Define an action of S_n on the basis vectors by $\sigma \cdot e_i = e_{\sigma(i)}$ for $\sigma \in S_n$. Extending the action linearly to all of V produces a linear action of S_n on V and thus a representation $X : S_n \to GL_n(F)$.

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- If G is cyclic of order m with generator g, define $X : G \to GL_1(\mathbb{C})$ by $X(g^i) = (\xi^i)$ where ξ is a fixed mth root of unity.
- Let G be a finite group and let N be a normal subgroup of G. Assume that N is isomorphic to a direct product of cyclic groups each of prime order p. Then N is a vector space over $\mathbb{F}_p$. There is a linear action of G on N induced by conjugation. In other words, conjugation by elements of G gives a representation $X : G \to GL(N) \cong GL_n(\mathbb{F}_p)$.

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For example, a typical element in $\mathbb{Q}S_3$ may be written

$$a_1 1 + a_2(1\,2) + a_3(1\,3) + a_4(2\,3) + a_5(1\,2\,3) + a_6(1\,3\,2)$$

Where all $a \in \mathbb{Q}$. An example of multiplication in $\mathbb{Q}S_3$:

$$[3(1\,2) - \frac{2}{3}(1\,2\,3)] \cdot [-(1\,2)] = -3 \cdot 1 + \frac{2}{3}(1\,3).$$

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We identify the elements $\lambda 1_G$ for $\lambda \in F$ with the field F. After making this identification we can say that $F \subseteq Z(FG)$.

Given an F-representation $X : G \to GL(V)$ of a finite group G we make V into a unital FG-module by defining

$$(\sum_{g \in G} a_g g) \cdot v = \sum_{g \in G} a_g X(g)(v).$$

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Conversely, if V is a unital FG-module then V is in particular an F-vector space (since $F \subseteq FG$) and restricting the "action" of FG to the basis G gives a linear action of G on V. Hence we get a representation.

We think of a representation in 3 ways:

(1) a homomorphism from a group into the group of nonsingular linear transformations of a vector space (or the group of nonsingular matrices)

(2) a linear action of a group on a vector space

(3) a module over a group algebra.

Since these notions all coincide we call each of them a representation.

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The third interpretation above provides a new example of a representation: the *regular representation* of G is FG considered as an FG-module (this is sometimes denoted ${}_{FG}FG$ to emphasize the left FG-module structure).

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Exercise: If X and Y are equivalent F-representations of G, show that $T^{-1}XT = Y$ for some linear transformation T.

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The "square-symmetry" representation $X : D_8 \to GL_2(\mathbb{R})$ is irreducible: otherwise, $\mathbb{R}^2$ would have an $\mathbb{R}D_8$ submodule of $\mathbb{R}$-dimension 1. But lines in $\mathbb{R}^2$ are not left invariant by the "rotation element" $r \in D_8$.

Definition

A representation $X : G \to GL(V)$ is *indecomposable* if V is an indecomposable FG-module (that is, V cannot be written as a direct sum $V = U \oplus W$ of FG-submodules in a nontrivial way).

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Let $Z_2 = \langle z \rangle$ and define $X : Z_2 \to GL_2(\mathbb{F}_2)$ by $X(z) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then X is a reducible, indecomposable representation: the subspace $\text{span}\{(1,0)\}$ is the only Z_2-invariant subspace of $\mathbb{F}_2^2$.

Maschke's Theorem

Let G be a finite group and F be a field whose characteristic does not divide the order of G. Let V be an FG-module and let U be an FG-submodule of V. Then there exists another FG-submodule W of V such that $V = U \oplus W$.

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$$\pi(hv) = \frac{1}{|G|} \sum_{g \in G} g\pi_0(g^{-1}hv) = \frac{1}{|G|} \sum_{g \in G} hh^{-1}g\pi_0((h^{-1}g)^{-1}v) = h\pi(v).$$

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It follows that if $u \in U \cap W$ then $0 = \pi(u) = u$, i.e., $U \cap W = 0$.If $v \in V$ then $\pi(v) \in U$ and $v - \pi(v) \in W$ since $\pi(v - \pi(v)) = \pi(v) - \pi(\pi(v)) = 0$. Thus $v = \pi(v) + (v - \pi(v)) \in U + W$ and $V = U \oplus W$. $\square$

Wedderburn's Theorem

Let R be a ring with $1 \neq 0$. The following are equivalent:

(1) Every R-module is injective.

(2) Every R-module is a direct sum of irreducible submodules.

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The matrix ring $M_n(D)$ over the division ring D has a unique isomorphism class of irreducible (left) modules. In essence, the only irreducible $M_n(D)$-module is D^n. It then follows directly from (3) above that FG has only *finitely many* irreducible modules!

If we choose our field F to be $\mathbb{C}$ then the division rings that appear in Wedderburn's Theorem all become $\mathbb{C}$. That is, for G a finite group,

$$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C}).$$

and $\mathbb{C}G$ has exactly r distinct isomorphism types of irreducible modules with complex dimensions $n_1, n_2, \ldots, n_r$.

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Note that

$$|G| = \dim_{\mathbb{C}}(\mathbb{C}G) = \sum_{i=1}^{r} \dim(M_{n_i}(\mathbb{C})) = \sum_{i=1}^{r} n_i^2.$$

It can be shown that each n_i divides $|G|$.

If we choose our field F to be $\mathbb{C}$ then the division rings that appear in Wedderburn's Theorem all become $\mathbb{C}$. That is, for G a finite group,

$$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C}).$$

and $\mathbb{C}G$ has exactly r distinct isomorphism types of irreducible modules with complex dimensions $n_1, n_2, \ldots, n_r$.

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Also note that $\dim_{\mathbb{C}}(Z(\mathbb{C}G)) = r$. It is not hard to show that a basis for $Z(\mathbb{C}G)$ is given by "class sums" $\sum_{g \in \mathcal{K}} g$ where $\mathcal{K}$ is a G-conjugacy class. It follows that r is the number of conjugacy classes of elements of G.

If A is an abelian group then A has $|A|$ conjugacy classes. Under the notation of the previous frame we have $r = |A|$ and this forces each $n_i = 1$. Thus all irreducible $\mathbb{C}A$-modules have dimension 1.

If G is non-abelian then $r < |G|$. Thus some $n_i > 1$. We have observed that a group G is abelian if and only if all irreducible $\mathbb{C}G$-modules have dimension 1 over $\mathbb{C}$.

Characters

We want to get our hands on the irreducible $\mathbb{C}$-representations of G. The problem is that, while there are only finitely many (up to equivalence), their degrees can be massive. For example, the monster group M is a simple group of order

$$\begin{aligned} |M| = {} & 808,017,424,794,512,875,886,459, \\ & 904,961,710,757,005,754,368,000,000,000. \end{aligned}$$

(According to Wikipedia, the order of M is approximately 808 sexdecillion.)

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(According to Wikipedia, the order of M is approximately 808 sexdecillion.) The smallest nontrivial irreducible representation of M

(famously) has degree $196,883$. A square matrix of order $196,883$ has $38,762,915,689$ entries.

Observe that if $X : G \to GL_n(\mathbb{C})$ is a representation and $g \in G$ has order k, then $X(g)^k = 1$. In other words the minimal polynomial of $X(g)$ divides $x^k - 1$, hence $X(g)$ is diagonalizable. This suggests studying the eigenvalues of $X(g)$ for the various elements of G and leads to the following definition:

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Note that $\chi(1) = \text{trace}(X(1)) = \text{trace}(I) = n$, the degree of the representation X.

Some examples:

- The character χ of the trivial representation $X : G \to GL_1(\mathbb{C})$ is just the constant function 1 on G: $\chi(g) = 1$ for all $g \in G$. This character is called the *principal character of* G.

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- The character χ of the trivial representation $X : G \to GL_1(\mathbb{C})$ is just the constant function 1 on G: $\chi(g) = 1$ for all $g \in G$. This character is called the *principal character of* G.
- Let G act on $\{1, 2, \ldots, n\}$ and let V be a $\mathbb{C}$-space with basis $e_1, e_2, \ldots, e_n$. Let G act on the basis vectors by $g \cdot e_i = e_{g \cdot i}$ and extend the action linearly to all of V. Then V is a $\mathbb{C}G$-module with associated homomorphism $X : G \to GL(V)$. The matrix $X(g)$ with respect to the basis $e_1, e_2, \ldots, e_n$ has a 1 in position i, j if and only if $g \cdot j = i$ and zeros elsewhere. In particular, there will be a 1 in position i, i if $g \cdot i = i$ and a 0 otherwise. This shows that, if χ is the character afforded by X, then

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- If ρ is the character of the regular representation of G then $\rho(1) = |G|$ and $\rho(g) = 0$ for all $g \neq 1$.

We adopt module-theoretic terminology to describe characters. For example, if the representation X is irreducible we say that the afforded character χ is irreducible and vice versa.

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If Q is an $n \times n$ matrix and P is an invertible $n \times n$ matrix then

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Note also that if $g, h \in G$ and X is a representation affording the character χ then

$$\begin{aligned}\chi(h^{-1}gh) = \operatorname{trace}(X(h^{-1}gh)) &= \operatorname{trace}(X(h)^{-1}X(g)X(h)) \\ &= \operatorname{trace}(X(g)) = \chi(g)\end{aligned}$$

so characters are constant on conjugacy classes.

If V is a (finite-dimensional) $\mathbb{C}G$-module by Wedderburn's Theorem we have

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_s$$

for some irreducible submodules U_i of V. Let χ be the character afforded by V and ψ_i the character afforded by U_i. Each element $g \in G$ then acts on V as a block-diagonal matrix with blocks corresponding to the modules U_i. It follows that

$$\chi = \psi_1 + \psi_2 + \cdots + \psi_s.$$

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It is not difficult to show that the irreducible characters of a group G are linearly independent. It follows that *two $\mathbb{C}$-representations are equivalent if and only if they afford the same character*.

This observation reduces the problem of studying equivalence classes of irreducible representations to the study of irreducible characters.

Definition

Let G be a finite group with r conjugacy classes. The *character table* of G is an $r \times r$ table whose columns are indexed by the conjugacy classes of G, whose rows are indexed by the irreducible characters of G, and whose entry in row $(\chi, \mathcal{K})$ is $\chi(g)$, g an element of the conjugacy class $\mathcal{K}$.

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For example, let $Z_2 = \langle z \rangle$. We have already observed that Z_2 has the trivial representation and the representation $X : Z_2 \to GL_1(\mathbb{C})$ defined by $X(z) = (-1)$. These representations are clearly inequivalent and since $|Z_2| = 2$ these are the only irreducible representations of Z_2. The character table of Z_2 is

	1	z
χ_1	1	1
χ_2	1	-1

One helpful observation in the determination of the character table of a group G is that if ρ denotes the character of the regular representation of G then

$$\rho = \sum_{\chi \text{ irreducible}} \chi(1)\chi.$$

This follows from the Wedderburn decomposition of $\mathbb{C}G$.

Let's see how this can be used by constructing the character table of S_3. The conjugacy classes of S_3 are represented by 1, $(1\,2)$, and $(1\,2\,3)$. Thus S_3 has 3 irreducible characters.

	1	$(1\,2)$	$(1\,2\,3)$
χ_1			
χ_2			
χ_3			

	1	$(1\,2)$	$(1\,2\,3)$
χ_1			
χ_2			
χ_3			

Any group has the principal character, and the group S_3 has the *sign homomorphism* which assigns 1 or -1 to a permutation depending on its cycle type. This can be viewed as a homomorphism from S_3 to $GL_1(\mathbb{C})$ so we have

	1	(1 2)	(1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3			

	1	(1 2)	(1 2 3)
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χ_3			

Now the equality $\rho = \chi_1 + \chi_2 + \chi_3(1)\chi_3$ allows us to fill in the final row:

$$6 = \rho(1) = \chi_1(1) + \chi_2(1) + \chi_3(1)^2 = 2 + \chi_3(1)^2,$$

thus $\chi_3(1) = 2$.

$0 = \rho((1\,2)) = \chi_1((1\,2)) + \chi_2((1\,2)) + \chi_3(1)\chi_3((1\,2)) = 1 + (-1) + 2\chi_3((1\,2))$,

thus $\chi_3((1\,2)) = 0$. Similarly, $\chi_3((1\,2\,3)) = -1$.

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Let π be the permutation character afforded by the representation of S_3 on a vector space of dimension 3 (so S_3 permutes the basis vectors around just as it permutes the numbers 1,2,3 around). We know that $\pi(g)$ is the number of fixed points of g on $\{1, 2, 3\}$. So $\pi(1) = 3$, $\pi((1\,2)) = 1$, and $\pi((1\,2\,3)) = 0$. One can see immediately that a decomposition of π is

$$\pi = \chi_1 + \chi_3.$$

Lots of information about a group G can be obtained by studying the character table of G. We have already seen, for example, that a group G is abelian if and only if all of its irreducible character degrees are 1.

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Let X be a representation that affords the character χ of G. Since X is a group homomorphism, $\ker(X)$ is a normal subgroup of G. Now $g \in \ker(X)$ if and only if $X(g)$ is the identity matrix, which holds if and only if $\chi(g) = \chi(1)$. Thus

$$\ker(\chi) = \{g \in G : \chi(g) = \chi(1)\}$$

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In particular, G is simple if and only if $\chi(g) = \chi(1)$ implies $g = 1$ for all irreducible characters χ of G.

The character table of G tells you...

- the order of G
- the number of conjugacy classes of G
- whether or not G is abelian, nilpotent, solvable, or simple
- $Z(G)$
- $|G : G'|$, the index of the commutator subgroup
- information about representations over fields of prime characteristic

The character table of G does *not* tell you the isomorphism type of G.
The character tables of D_8 and Q_8 are identical but $D_8 \ncong Q_8$.

Two of the main tools in constructing a character table are...

- The Hermitian inner product $(\chi, \psi) = \dfrac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1})$ for characters χ and ψ of G. This inner product satisfies $(\chi, \psi) = \delta_{\chi\psi}$ for irreducible characters χ, ψ of G and allows one to perform fast decompositions of reducible characters.

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- Induction: If $H \leq G$ and V is a $\mathbb{C}H$-module, then $\mathbb{C}G \otimes_{\mathbb{C}H} V$ is a $\mathbb{C}G$ module. This tool allows one to build characters of groups using characters of subgroups.

Character theory is a wonderful tool for proving theorems about groups. For example...

Burnside's $p^a q^b$ Theorem

A group of order $p^a q^b$ where p, q are primes is necessarily solvable.

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Frobenius

Let G be a finite group and H be a nontrivial proper subgroup of G such that $H \cap H^g = 1$ for all $g \in G \setminus H$. Then there exists a normal subgroup N of G such that $NH = G$ and $N \cap H = 1$.